Minimal Bad Sequence on Quasi-Orders

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1 Description

In [2] we presented the new JRA's instantiation context to define closure, fixpoint (Tarski), well-founded (Noether) and recursion. A new instantiation plugin [6] was developed in the EBRP project [7]. In this paper we present quasi-orders and well-quasi order and two important theorems on quasi-orders: the existence of a minimal bad sequences when the quasi-order is well-founded but not a well-quasi-order and the second theorem: the set of all values strictly less then value of the minimal bad sequence is well-quasi oder. Rodin [8] is used to develop and prove all theorems describe in this paper.

2 Definitions and some theorems

Let S_type a carrier set A quasi-order is a reflexive and transitive relation.

$qo = \{S \mapsto g | S \subseteq S\_type \wedge g \in S \leftrightarrow S \wedge g; g \subseteq g \wedge S \lhd id \subseteq g\}$
$wqo = \{S \mapsto g | S \mapsto g \in qo \wedge (\forall f \cdot f \in \mathbb{N} \rightarrow S \Rightarrow (\exists i, j \cdot i \geq 0 \wedge j > i \wedge f(i) \mapsto f(j) \in g))\}$
$sdc = (\lambda S \mapsto g . S \mapsto g \in qo | \{f | f \in \mathbb{N} \rightarrow S \wedge (\forall i, j \cdot i \geq 0 \wedge j > i \Rightarrow f(j) \mapsto f(i) \in g \setminus g^{-1})\})$
$wf = \{S \mapsto g | S \mapsto g \in qo \wedge sdc(S \mapsto g) = \varnothing\}$
$antichain = (\lambda S \mapsto g . S \mapsto g \in qo | \{A | A \subseteq S \wedge (A \times A) \cap g \subseteq id\})$

We have proved much theorems till Kruskal's one given in [3] like the Lemma1.3.2
$\forall S, g, A \cdot S \mapsto g \in wf \wedge A \subseteq S \Rightarrow$
$(\exists A_0 \cdot A_0 \in antichain(S \mapsto g) \wedge A_0 \subseteq A \wedge (\forall x \cdot x \in A \Rightarrow (\exists a \cdot a \in A_0 \wedge a \mapsto x \in g)))$.
We have used it to prove the following one (reformulation of the Lemma1.3.1 (existence of a minimum).
$\forall S, g, A \cdot S \mapsto g \in wf \wedge A \in \mathbb{P}1(S) \Rightarrow (\exists m \cdot m \in A \wedge (\forall z \cdot z \in A \wedge z \mapsto m \in g \Rightarrow m \mapsto z \in g))$
We have used our $FrSB$ operator [2] to define general recursive function from $\mathbb{N}$ to S_type First we instantiate $FrSB$ with $S, B := \mathbb{N}, S\_type$. $\{i \mapsto j | i \geq 0 \wedge i < j\}$ is a well-founded relation on $\mathbb{N}$. Let g be a a function such that: $g \in (\mathbb{N} \times (\mathbb{N} \nrightarrow S\_type)) \rightarrow S\_type$ .There is a unique total function fr: $fr \in \mathbb{N} \rightarrow S\_type$ such that we have: $\forall n \cdot n \in \mathbb{N} \Rightarrow fr(n) = g(n \mapsto 0..n-1 \lhd fr)$ The value of fr at n depends on its value on the set $0..n-1$, $FrSB$ is a function (an operator) which gives the recursive fonction fr: $fr = FrSB(\{i \mapsto j | i \geq 0 \wedge i < j\} \mapsto g)$

3 Bad sequence

Let $S \mapsto g$ be in qo a bad sequence bs is a function from $\mathbb{N}$ to S where $\forall i, j \cdot i \geq 0 \wedge j > i \Rightarrow bs(i) \mapsto bs(j) \notin g$.
Remark: if $S \mapsto g$ not in wqo the set of bad sequence is not empty. Let BS be the set of bad sequence on S.
bs is minimal if
$\forall n, f \cdot n \geq 0 \wedge f \in \mathbb{N} \rightarrow S \Rightarrow \wedge 0..n-1 \lhd f = 0..n-1 \lhd bs \wedge f(n) \mapsto bs(n) \in g \setminus g^{-1}$
$\Rightarrow$
$(\exists i, j \cdot i \geq 0 \wedge j > i \wedge f(i) \mapsto f(j) \in g))$

3.1 Existence of a minimal bad sequence

When a qo is wf but not wqo a minimal bad sequence exists [4]. Let ch be the choice function on S_type: $ch \in \mathbb{P}1(S\_type) \to S\_type$ and $\forall s \cdot s \in \mathbb{P}1(S\_type) \Rightarrow ch(s) \in s$. Let $chmin$ be the function which give a minimum in a non empty set when the quasi-order is in wf we have:

$chmin = (\lambda A \cdot A \in \mathbb{P}1(S) | ch(\{m | m \in A \wedge (\forall z \cdot z \in A \wedge z \mapsto m \in g \Rightarrow m \mapsto z \in g)\})$

We instantiate g in $FrSB$ with

$$
\begin{array}{l}
\{n, k, b \cdot n \geq 0 \wedge k \in \mathbb{N} \nrightarrow S\_type \wedge 0..n-1 \subseteq \mathrm{dom}(k) \wedge \\
\quad (\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = k(i)) | f(n)\} \neq \varnothing \\
\quad\quad \Rightarrow b = chmin(\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = k(i)) | f(n)\})) \wedge \\
\quad (\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = k(i)) | f(n)\} = \varnothing \Rightarrow b = ch(S\_type)) \\
\mid\ n \mapsto k \mapsto b\}.
\end{array}
$$

let bs be the sequence $FrSB(\{i \mapsto j | i \geq 0 \wedge i < j\} \mapsto g)$ with our new g we got for free $bs \in \mathbb{N} \to S\_type$ and $\forall n \cdot n \in \mathbb{N} \ \Rightarrow\ bs(n) = g(n \mapsto 0..n-1 \lhd bs)$ then we have

$\forall n \cdot n \in \mathbb{N} \ \wedge\ \{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\} \neq \varnothing$

$\quad \Rightarrow\ bs(n) = chmin(\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\})$ and

$\forall n \cdot n \in \mathbb{N} \ \wedge\ \{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\} = \varnothing$

$\quad \Rightarrow\ bs(n) = ch(S\_type)$

Now we can prove by recurrence on n that $\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\} \neq \varnothing$ and then we can prove that $\forall n \cdot n \in \mathbb{N} \Rightarrow bs(n) = chmin(\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\})$ and $\forall n \cdot n \in \mathbb{N} \ \Rightarrow\ bs(n) \in S$.

We can conclude that bs is a minimal bad sequence.

3.2 A well-quasi-order under value of a minimal bad sequence

When a qo is wf but not wqo and bs a bad sequence then $(g^{-1} \setminus g)[ran(bs)] \mapsto ((g^{-1} \setminus g)[ran(bs)] \lhd g \rhd (g^{-1} \setminus g)[ran(bs)] \in wqo$. To prove this theorem we have follow the proof of the lemma 22 in [5].

4 Conclusion

This two lemmas was not well defined but used in [3]. With both we have proved more easily but in the same way the Higman's lemma and the Kruskal's theorem.

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