Minimal Bad Sequence on Quasi-Orders

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June 2025

- End of november Thierry Coquand send me a paper on ordinals where Kruskal's theorem is mentioned. I never hear on quasi-orders. To train I've manage some proof on total order.

- At the beginning of december I've found ENS Cachan lecture on this topic where proofs are well explained

  S. Demri, A. Finkel, J. Goubault-Larrecq, S. Schmitz and PH. Schnoebelen. *Well-Quasi-Orders For Algorithms MPRI Course 2.9.1 -2017/2018*.

- Strangely theorems on minimal bad seqence are not present but used in Cachan lecture

- 4 weeks proof effort

- We have used our general recursive operator $\boldsymbol{FrSB}$ to construct sequences.

Definitions

Let S_type be a carrier set.

A quasi-order is a reflexive and transitive relation.

$$qo = \{S \mapsto g | S \subseteq S\_type \wedge g \in S \leftrightarrow S \wedge S \lhd id \subseteq g \wedge g ; g \subseteq g\}$$

In a well-quasi-order all infinite sequences are good.

$$\begin{aligned} wqo = \{S \mapsto g | S \mapsto g \in qo \wedge \\ (\forall f \cdot f \in \mathbb{N} \rightarrow S \\ \Rightarrow \\ (\exists i, j \cdot i \geq 0 \wedge j > i \wedge f(i) \mapsto f(j) \in g))\} \end{aligned}$$

Definitions

$$\begin{aligned} sdc = (\lambda S \mapsto g \cdot S \mapsto g \in qo \mid \\ \{f | f \in \mathbb{N} \to S \wedge \\ (\forall i, j \cdot i \geq 0 \wedge j > i \Rightarrow f(j) \mapsto f(i) \in g \setminus g^{-1})\}) \end{aligned}$$

$$wf = \{S \mapsto g | S \mapsto g \in qo \wedge sdc(S \mapsto g) = \varnothing\}$$

$$antichain =$$

$$(\lambda S \mapsto g \ \cdot \ S \mapsto g \in qo | \{A | A \subseteq S \wedge (A \times A) \cap g \subseteq id\})$$

We assume the Axiom of Choice using a choice function. Remark in a quasi order two element can be equivalent then we can define the set of classes of a qo.

$$class =$$
$$(\lambda S \mapsto g . S \mapsto g \in qo |$$
$$\cup x \cdot x \in S | \{\{y | y \in S \wedge x \mapsto y \in g \wedge y \mapsto x \in g\}\})$$

and using the choice function ch on S_type ($ch \in \mathbb{P}1(S\_type) \rightarrow S\_type$ and $\forall s \cdot s \in \mathbb{P}1(S\_type) \Rightarrow ch(s) \in s$) we can have the set of canonical representatives of classes.

$$crclass = (\lambda S \mapsto g . S \mapsto g \in qo | ch[class(S \mapsto g)])$$

We instantiate $FrSB$ with $S, B := \mathbb{N}, S\_type$.

r is a well-founded relation on $\mathbb{N}$. Let g be a function such that: $g \in (\mathbb{N} \times (\mathbb{N} \nrightarrow S\_type)) \rightarrow S\_type$.

There is a unique total function fr: $fr \in \mathbb{N} \rightarrow S\_type$ such that we have:

$$\forall n \cdot n \in \mathbb{N} \;\Rightarrow\; fr(n) = g(n \mapsto r^{-1}[\{n\}] \lhd fr)$$

The value of fr at n depends on its value on the set $r^{-1}[\{n\}]$, $FrSB$ is a function (an operator) which gives the recursive fonction fr: $fr \;=\; FrSB(r \mapsto g)$

We instantiate $FrSB$ with $S, B := \mathbb{N}, S\_type$.

$\{i \mapsto j | i \geq 0 \land i < j\}$ is a well-founded relation on $\mathbb{N}$. Let g be a function such that: $g \in (\mathbb{N} \times (\mathbb{N} \nrightarrow S\_type)) \rightarrow S\_type$.

There is a unique total function fr: $fr \in \mathbb{N} \rightarrow S\_type$ such that we have: ($\{i \mapsto j | i \geq 0 \land i < j\}^{-1}[\{n\}] = 0..n-1$)

$$\forall n \cdot n \in \mathbb{N} \;\Rightarrow\; fr(n) = g(n \mapsto 0..n-1 \lhd fr)$$

The value of fr at n depends on its value on the set $0..n-1$, $FrSB$ is a function (an operator) which gives the recursive fonction fr: $fr \;=\; FrSB(\{i \mapsto j | i \geq 0 \land i < j\} \mapsto g)$

If we instantiate $FrSB1$ with $S, B := \mathbb{N}, S\_type$.

$\{i \mapsto i+1 | i \geq 0\}$ is a well-founded relation on $\mathbb{N}$.

($\{i \mapsto i+1 | i \geq 0\}^{-1}(n) = n-1$)

If $fr \;=\; FrSB1(\{i \mapsto i+1 | i \geq 0\} \mapsto f0 \mapsto f)$ we have

$$\mathrm{f}r(0) = f0(0) \text{ and}$$

$$\forall n \cdot n \geq 0 \;\Rightarrow\; fr(n+1) = f(n+1 \mapsto fr(n))$$

using $FrSB1$ we define $FrNB$ where

$FrNB(f0 \mapsto f) = FrSB(\{i \mapsto i+1 | i \geq 0\} \mapsto f0 \mapsto f)$

Convention

- Name of lemmas theorems are Cachan's one
- I'll not present all slides but deep explanations can be read after (see Rodin workshop webpage)
- If I say less, greater it's on the relation of a quasi-order (often g) except on $\mathbb{N}$

Lemma1.3.1

If $S \mapsto g$ is wf and $A \subseteq S$ then for all a in A there exists a minimum x of a

$$
\begin{aligned}
&\forall S, g, A \cdot S \mapsto g \in wf \wedge A \subseteq S \Rightarrow \\
&\qquad (\forall a \cdot a \in A \\
&\qquad\quad \Rightarrow \\
&\qquad\qquad (\exists x \cdot x \in A \wedge x \mapsto a \in g \wedge \\
&\qquad\qquad\qquad (\forall z \cdot z \in A \wedge z \mapsto x \in g \Rightarrow x \mapsto z \in g)))
\end{aligned}
$$

x is less than a and if z is less than x there are equivalent

Lemma1.3.1: Flavour of the proof

Proof: we define the following sequence

$fr(0)$	$\cdots$	$fr(n)$	$fr(n+1)$	$\cdots$

where $fr(0) = a$ and $\mathrm{ran}(fr) \subseteq A$ and

fr is a descending sequence

if two consecutive elements are equals then the sequence is stationary.

$FrNB(\{0 \mapsto a\} \mapsto f)$ where f is

$$(\lambda n \mapsto s \cdot n > 0 \wedge (\exists y \cdot y \in A \wedge y \mapsto s \in g \wedge s \mapsto y \notin g)|$$

$$ch(\{y|y \in A \wedge y \mapsto s \in g \wedge s \mapsto y \notin g\}))\cup$$

$$(\lambda n \mapsto s \cdot n > 0 \wedge (\forall y \cdot y \notin A \vee y \mapsto s \notin g \vee s \mapsto y \in g)|s)$$

On fr we prove that $\mathrm{ran}(fr) \subseteq A$ and fr is a descending sequence and if two consecutive elements are equals then the sequence is stationary. This sequence cannot always strictly decrease so it is stationary at a minimal element of a

Lemma1.3.2

$$\forall S, g, A \cdot S \mapsto g \in wf \wedge A \subseteq S \Rightarrow$$
$$(\exists A_0 \cdot A_0 \in antichain(S \mapsto g) \wedge A_0 \subseteq A \wedge$$
$$\forall x \cdot x \in A \Rightarrow (\exists a \cdot a \in A_0 \wedge a \mapsto x \in g))$$

Proof: let h be $\{x \mapsto y | x \in A \wedge y \in crclass(S \mapsto g) \wedge x \mapsto y \in g \wedge y \mapsto x \in g\}$, h is a function from A to $A \cap crclass(S \mapsto g)$

the witness for A_0 is

$$h[\{x | x \in A \wedge (\forall z \cdot z \in A \wedge z \mapsto x \in g \Rightarrow x \mapsto z \in g)\}]$$

We have used the previous lemma to prove the following one (reformulation of the Lemma1.3.1 : existence of a minimum).

$$\forall S, g, A \cdot S \mapsto g \in wf \wedge A \in \mathbb{P}1(S) \Rightarrow (\exists m \cdot m \in A \wedge (\forall z \cdot z \in A \wedge z \mapsto m \in g \Rightarrow m \mapsto z \in g))$$

This lemma is not present in Cachan lecture but is trivial we choose the minimum in the previous $antichain$ using the choice function ch: $chmin =$

$$(\lambda A \cdot A \in \mathbb{P}1(S) | ch(\{m | m \in A \wedge (\forall z \cdot z \in A \wedge z \mapsto m \in g \Rightarrow m \mapsto z \in g)\})$$

Theorem 1.5 (Characterisations of wqo)

Let $S \mapsto g$ be a qo the following are equivalent:

1. $S \mapsto g \in wqo$

2. $\forall f \cdot f \in \mathbb{N} \to S \Rightarrow$

   $(\exists X \cdot X \subseteq \mathbb{N} \wedge \neg finite(X) \wedge$

   $(\forall i, j \cdot i \in X \wedge j \in X \wedge i < j \Rightarrow f(i) \mapsto f(j) \in g))$

3. $S \mapsto g \in wf$ and $\forall A \cdot A \in antichain(S \mapsto g) \Rightarrow finite(A)$

Proof: $(2) \Rightarrow (1)$ is trivial $(1) \Rightarrow (3)$ is easy and $(3) \Rightarrow (2)$ is more difficult. Many people use Ramsey's theorem to prove this implication. In Cachan authors propose an other interesting proof. To manage this proof with the help of Cachan we instantiate in $\boldsymbol{FrNB}$ $\boldsymbol{B}$ by $\mathbb{N}$ to obtain the operator (function) $\boldsymbol{FrNN}$ and use the choice function $\boldsymbol{cN}$ ($\boldsymbol{cN} \in \mathbb{P}\,\mathbf{1}(\mathbb{N}) \rightarrow \mathbb{N}$ and $\forall \boldsymbol{s} \cdot \boldsymbol{S} \in \mathbb{P}\,\mathbf{1}(\mathbb{N}) \Rightarrow \boldsymbol{ch}(\boldsymbol{s}) \in \boldsymbol{s}$). Since all $\boldsymbol{antichain}$ are finite there exists a $\boldsymbol{k}$ that ($\forall \boldsymbol{x} \cdot \boldsymbol{x} \geq \boldsymbol{k} \Rightarrow (\exists \boldsymbol{j} \cdot \boldsymbol{j} > \boldsymbol{x} \wedge \boldsymbol{f}(\boldsymbol{x}) \mapsto \boldsymbol{f}(\boldsymbol{j}) \in \boldsymbol{g})$) We can define the following sequence $\boldsymbol{fr}$ by

$\boldsymbol{FrNN}(\{\mathbf{0} \mapsto \boldsymbol{k}\} \mapsto \boldsymbol{f})$ where $\boldsymbol{f}$ is

$$(\lambda n \mapsto s \cdot n > 0 \wedge s \geq k$$
$$|cN(\{j|j > s \wedge f(s) \mapsto f(j) \in g\}))\cup$$
$$(\lambda n \mapsto s \cdot n > 0 \wedge s \in 0..k - 1|s + 1)$$

Then we prove by recurrence that $(\forall n \cdot n \geq 0 \Rightarrow fr(n) \geq k)$,

$(\forall n \cdot n \geq 0 \Rightarrow fr(n) < fr(n + 1))$ and

$(\forall n \cdot n \geq 0 \Rightarrow (\forall x \cdot x \in 0..n \Rightarrow f(fr(x)) \mapsto f(fr(n)) \in g)))$

X is $\mathrm{ran}(fr)$

Bad sequence

- When $S \mapsto g$ is well-quasi-ordered then all infinite sequence are good

- f is a good sequence if

$$\exists i, j \cdot i \geq 0 \wedge j > i \wedge f(i) \mapsto f(j) \in g$$

- a bad sequence is not good

$$\forall i, j \cdot i \geq 0 \wedge j > i \Rightarrow f(i) \mapsto f(j) \notin g$$

Remark: if $S \mapsto g$ is qo but not in wqo the set of bad sequence is not empty. Let BS be the set of bad sequence on S.

bs is minimal if

$$\forall n, f \cdot n \geq 0 \wedge f \in \mathbb{N} \rightarrow S \Rightarrow \wedge 0..n-1 \lhd f = 0..n-1 \lhd bs \wedge f(n) \mapsto bs(n) \in g \setminus g^{-1}$$

$$\Rightarrow$$

$$(\exists i, j \cdot i \geq 0 \wedge j > i \wedge f(i) \mapsto f(j) \in g))$$

- Stangely the existence of the minimal bad sequence is not defined and proved in Cachan lecture but the argument of the minimal bad sequence is well used.

- When a qo is wf but not wqo a minimal bad sequence exists (Nash-Williams).

- Let $chmin$ be the function which give a minimum in a non empty set when the quasi-order is in wf we have:

  $chmin = (\lambda A \cdot A \in \mathbb{P}1(S)|ch(\{m|m \in A \wedge (\forall z \cdot z \in A \wedge z \mapsto m \in g \Rightarrow m \mapsto z \in g)\})$

Existence of a minimal bad sequence

bs is define step by step. we take all f in BS with the same prefix than bs in construction.

$$bs(n) =$$
$$chmin(\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\})$$

We can conclude that bs is a minimal bad sequence.

We need to prove that

$$\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\} \neq \varnothing$$

We instantiate g in $FrSB$ with

$$
\begin{array}{l}
\{n, k, b \cdot n \geq 0 \land k \in \mathbb{N} \nrightarrow S\_type \land 0..n-1 \subseteq \mathrm{dom}(k) \land \\
\quad (\{f \cdot f \in BS \land (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = k(i)) | f(n)\} \neq \varnothing \\
\quad\ \Rightarrow \\
\qquad b = chmin(\{f \cdot f \in BS \land \\
\qquad\qquad\qquad (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = k(i)) | f(n)\})) \land \\
\quad (\{f \cdot f \in BS \land (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = k(i)) | f(n)\} = \varnothing \\
\quad\ \Rightarrow \\
\qquad b = ch(S\_type)) \\
\quad |\ n \mapsto k \mapsto b\}.
\end{array}
$$

let bs be the sequence $FrSB(\{i \mapsto j | i \geq 0 \land i < j\} \mapsto g)$ with our new g we got for free

$bs \in \mathbb{N} \rightarrow S\_type$ and $\forall n \cdot n \in \mathbb{N} \ \Rightarrow \ bs(n) = g(n \mapsto 0..n-1 \lhd bs)$ then we have

Existence of a minimal bad sequence

$$\forall n \cdot n \in \mathbb{N} \;\wedge\; \{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\} \neq \varnothing \;\Rightarrow\; bs(n) = chmin(\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\})$$

Now we can prove by recurrence on n that

$$\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\} \neq \varnothing$$

and then we can prove that

$\forall n \cdot n \in \mathbb{N} \Rightarrow$
$bs(n) = chmin(\{f \cdot f \in BS \wedge (\forall i \cdot i \in 0..n-1 \Rightarrow f(i) = bs(i)) | f(n)\})$
and $\forall n \cdot n \in \mathbb{N} \;\Rightarrow\; bs(n) \in S$.

We can conclude that bs is a minimal bad sequence.

When a $\boldsymbol{qo}$ is $\boldsymbol{wf}$ but not $\boldsymbol{wqo}$ and $\boldsymbol{bs}$ a minimal bad sequence then

$$(\boldsymbol{g}^{-1} \setminus \boldsymbol{g})[\mathbf{ran}(\boldsymbol{bs})]$$
$$\mapsto$$
$$((\boldsymbol{g}^{-1} \setminus \boldsymbol{g})[\mathbf{ran}(\boldsymbol{bs})] \lhd \boldsymbol{g} \rhd (\boldsymbol{g}^{-1} \setminus \boldsymbol{g})[\mathbf{ran}(\boldsymbol{bs})]) \in \boldsymbol{wqo}.$$

To prove this theorem we have followed the proof of the lemma 22 (Laszlo Székely and Éva Czabarka.).

Then $\forall \boldsymbol{A} \cdot \boldsymbol{A} \subseteq (\boldsymbol{g}^{-1} \setminus \boldsymbol{g})[\mathbf{ran}(\boldsymbol{bs})] \Rightarrow \boldsymbol{A} \mapsto (\boldsymbol{A} \lhd \boldsymbol{g} \rhd \boldsymbol{A}) \in \boldsymbol{wqo}$

Lemma 22. *Assume that $(A, \trianglelefteq)$ is well-founded quasi-order, in which $x_0, x_1, x_2, \ldots$ is a minimal bad sequence. Consider*

$$Y = \left\{ x \in A : \exists i \text{ such that } x \triangleleft x_i \right\}$$

(note the strict inequality in the formula!). Then $Y = \emptyset$ or $(Y, \trianglelefteq |_Y)$ is a WQO.

Proof. If $Y \neq \emptyset$ and $(Y, \trianglelefteq |_Y)$ is not a WQO, then, based on Lemma 19, select a minimal bad sequence $y_0, y_1, y_2, \ldots$ for $(Y, \trianglelefteq |_Y)$. By the construction of Y,

$$\forall i \geq 0 \, \exists i' \geq 0 \text{ such that } y_i \triangleleft x_{i'}.$$

Select the smallest natural number i' that comes up in this way, and select the smallest index i that produces this i'. Now $y_i, y_{i+1}, y_{i+2}, \ldots$ is still a bad sequence for $(Y, \trianglelefteq |_Y)$, as it is an infinite subsequence of a bad sequence. By part (iii) of the Proposition above, $y_i, y_{i+1}, y_{i+2}, \ldots$ is also a bad sequence in $(A, \trianglelefteq)$. If $i' = 0$, the bad sequence $y_i, y_{i+1}, y_{i+2}, \ldots$ contradicts the minimality of the bad sequence $x_0, x_1, x_2, \ldots$ by $y_i \triangleleft x_0$.
Assume now $i' \geq 1$. We are going to show that

$$x_0, x_1, \ldots, x_{i'-1}, y_i, y_{i+1}, y_{i+2}, \ldots$$

is a bad sequence in $(A, \trianglelefteq)$, which will contradict the minimality of the bad sequence $x_0, x_1, x_2, \ldots$, as $y_i \triangleleft x_{i'}$. Indeed, no pair in the x part of the sequence and no pair in the y part of the sequence would fail badness. The only possible problem is if there is an x_m with $m \leq i' - 1$ and y_n with $n \geq i$ such that $x_m \trianglelefteq y_n$. As $y_n \in Y$, there is an n' such that $y_n \triangleleft x_{n'}$. By the choice of i', $m \leq i' - 1 < i' \leq n'$ and $x_m \trianglelefteq x_{n'}$, contradicting the badness of the $x_0, x_1, x_2, \ldots$ sequence. $\square$

Exemple

In the following slides we have used the instantiation plugin (EBRP) on our quasi-order context

- Cartesian Product
- Cartesian product of a finite family of quasi-orders
- Finite Words: Higmann's Lemma

Let A_type and B_type two carrier sets. By three instantiations :

1. $S\_type, qo, wqo := A\_type, qoA, wqoA$,

2. $S\_type, qo, wqo := B\_type, qoB, wqoB$

3. $S\_type, qo, wqo := A\_type \times B\_type, qoAxB, wqoAxB$

we can obtain for free (above all $wqoA$)

$$qoA = \{S \mapsto g | S \subseteq A\_type \wedge \ldots\}$$

$$wqoA = \{S \mapsto g | S \mapsto g \in qoA \wedge \ldots \exists X \ldots\}$$

$$qoB = \{S \mapsto g | S \subseteq B\_type \wedge \ldots\}$$

$$wqoB = \{S \mapsto g | S \mapsto g \in qoB \wedge \ldots \exists i, j \ldots\}$$

$$qoAxB = \{S \mapsto g | S \subseteq A\_type \times B\_type \wedge \ldots\}$$

$$wqoAxB = \{S \mapsto g | S \mapsto g \in qoAxB \wedge \ldots \exists i, j \ldots\}$$

then we can prove the two following theorems:

$$\begin{aligned} &\forall A, g, B, k \cdot A \mapsto g \in qoA \wedge B \mapsto k \in qoB \Rightarrow \\ &\qquad (A \times B) \mapsto \\ &\qquad \{(a \mapsto b) \mapsto (a' \mapsto b') | a \mapsto a' \in g \wedge b \mapsto b' \in k\} \in qoAxB \end{aligned}$$

$$\begin{aligned} &\forall A, g, B, k \cdot A \mapsto g \in wqoA \wedge B \mapsto k \in wqoB \Rightarrow \\ &\qquad (A \times B) \mapsto \\ &\qquad \{(a \mapsto b) \mapsto (a' \mapsto b') | a \mapsto a' \in g \wedge b \mapsto b' \in k\} \in wqoAxB \end{aligned}$$

The first one is trivial. For the second on the sequence of $A \times B$ the A part is a sequence on A then there is a X where the sequence is monotone. If we restrict the sequence on X for the B part we have an i and j where the sequence (on B) is monotone then monotone on $A \times B$.

Cartesian Product: Compromise:

With Rodin we cannot define a family of carrier set then if we want to define $\prod_{i=1}^{n} D_i$ where all D_i are qo (or wqo) all D_i need to have the same carrier set. we instantiate quasi-order with S_type then with $\mathbb{P}(\mathbb{Z} \times S\_type)$ to obtain

$$qo = \{S \mapsto g | S \subseteq S\_type \land \ldots\}$$

$$wqo = \{S \mapsto g | S \mapsto g \in qo \land \ldots \exists i, j \ldots\}$$

$$wqo = \{S \mapsto g | S \mapsto g \in qo \land \ldots \exists X \ldots)\}$$

$$qoP = \{S \mapsto g | S \subseteq \mathbb{P}(\mathbb{Z} \times S\_type) \land \ldots\}$$

$$wqoP = \{S \mapsto g | S \mapsto g \in qoP \land \ldots \exists i, j \ldots\})$$

Cartesian Product: Compromise:

$$
\begin{array}{l}
\forall n, PS, Pg . n \geq 0 \wedge \\
\qquad PS \in 1..n \rightarrow \mathbb{P}(S\_type) \wedge Pg \in 1..n \rightarrow (S\_type \leftrightarrow S\_type) \wedge \\
\qquad (\forall i \cdot i \in 1..n \Rightarrow PS(i) \mapsto Pg(i) \in wqo) \\
\Rightarrow \\
\qquad \{S | S \in 1..n \rightarrow S\_type \wedge (\forall i \cdot i \in 1..n \Rightarrow S(i) \in PS(i))\} \\
\mapsto \\
\qquad \{S \mapsto S' | S \in 1..n \rightarrow S\_type \wedge (\forall i \cdot i \in 1..n \Rightarrow S(i) \in PS(i)) \wedge \\
\qquad\qquad S' \in 1..n \rightarrow S\_type \wedge (\forall i \cdot i \in 1..n \Rightarrow S'(i) \in PS(i)) \wedge \\
\qquad\qquad (\forall i \cdot i \in 1..n \Rightarrow S(i) \mapsto S'(i) \in Pg(i))\} \in wqoP
\end{array}
$$

n subsets $PS(i)$ n relations $Pg(i)$

n $wqos$ $PS(i) \mapsto Pg(i)$

Cartesian Product: Compromise:

Then we can use the previous theorem to prove the corollary 1.18 (before 1.17) where all $PS(i)$ are the same

$$
\begin{array}{l}
\forall n, X, Pg . n \geq 0 \wedge \\
\qquad X \subseteq S\_type \wedge Pg \in 1..n \rightarrow (X \leftrightarrow X) \wedge \\
\qquad (\forall i \cdot i \in 1..n \Rightarrow X \mapsto Pg(i) \in wqo) \\
\quad \Rightarrow \\
\qquad \{S \mid S \in 1..n \rightarrow X\} \\
\quad \mapsto \\
\qquad \{S \mapsto S' \mid S \in 1..n \rightarrow X \wedge S' \in 1..n \rightarrow X \wedge \\
\qquad\qquad (\forall i \cdot i \in 1..n \Rightarrow S(i) \mapsto S'(i) \in Pg(i)))\} \in wqoP
\end{array}
$$

Cartesian Product: Compromise:

Now we can use the previous theorem to prove an abstraction of corollary 1.17 (Dikson's lemma)

$$\begin{array}{l} \forall n, X, g . n \geq 0 \wedge \\ \quad X \mapsto g \in wqo \\ \Rightarrow \\ \quad \{S | S \in 1..n \rightarrow X\} \\ \mapsto \\ \quad \{S \mapsto S' | S \in 1..n \rightarrow X \wedge S' \in 1..n \rightarrow X \wedge \\ \qquad (\forall i \cdot i \in 1..n \Rightarrow S(i) \mapsto S'(i) \in g))\} \in wqoP \end{array}$$

To prove the Dickson's lemma $\mathbb{N}^n$ is wqo (1913), we instantiate the previous with $S\_type := \mathbb{Z}$ and $X, g := \mathbb{N}, \leq$

$$\begin{array}{l} \forall n \cdot n \geq 0 \Rightarrow \\ \quad \{S | S \in 1..n \rightarrow \mathbb{N}\} \\ \mapsto \\ \quad \{S \mapsto S' | S \in 1..n \rightarrow \mathbb{N} \wedge S' \in 1..n \rightarrow \mathbb{N} \wedge \\ \qquad (\forall i \cdot i \in 1..n \Rightarrow S(i) \leq S'(i))\} \in wqoP \end{array}$$

Finite words: Higman's Lemma

Let A (typed by A_type) be a set of the alphabet of words. A finite words is a finite sequence:

$$Words = \{w | \exists n \cdot n \geq 0 \wedge w \in 1..n \rightarrow A\}.$$

In Cachan lecture they use an inductive definition (3 rules then 3 cases) for the order. Here we use an growing injection p

w $\begin{array}{|c|c|c|} \hline w(1) & w(2) & w(3) \\ \hline \end{array}$ w' $\begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & w'(p(1)) & \cdot & w'(p(2)) & \cdot & w'(p(3)) & \cdot \\ \hline \end{array}$

$w(1) \mapsto w'(p(1)) \in g$ $w(2) \mapsto w'(p(2)) \in g$

$w(3) \mapsto w'(p(3)) \in g$

If g is a quasi-order on A the the "quasi-order" on words is defined by the following relation:

$$
\begin{aligned}
&RelW = \\
&\{w \mapsto w' | \exists n, m \cdot n \geq 0 \land w \in 1..n \rightarrow A \land \\
&\qquad m \geq 0 \land x' \in 1..m \rightarrow A \land \\
&\qquad (\exists p \cdot p \in 1..n \rightarrowtail 1..m \land \\
&\qquad\quad (\forall i, j \cdot i \geq 1 \land j \in i + 1..n \\
&\qquad\quad \Rightarrow p(i) < p(j)) \land \\
&\qquad\quad (\forall i \cdot i \in 1..n \Rightarrow w(i) \mapsto w'(p(i)) \in g)\}
\end{aligned}
$$

It's an embedding qo on words.

Let A_type a carrier set. By two instantiations :

1. $S\_type, qo, wqo, sdc, wf \ := A\_type, qo, wqo, sdc, wf$,

2. $S\_type, qo, wqo, sdc, wf \ :=$
   $\mathbb{P}(\mathbb{Z} \times A\_type), qoW, wqoW, sdcW, wfW$

Higman's lemma is:

$$\forall A, g \cdot A \mapsto g \in wqo \Rightarrow Words \mapsto RelW \in wqoW$$

We instantiate cartesian product with:

$$qoA, wqoA, B\_type, qoB, wqoB, qoAxB, wqoAxB :=$$
$$qo, wqo, \mathbb{P}(\mathbb{Z} \times A\_type), qoW, wqoW, qoxW, wqoxW$$

Remark we instantiate also with the corresponding sdc and wf

Proof: By contradiction. We assume $Words \mapsto RelW \notin wqoW$. $Words \mapsto RelW \in qoW$ because $A \mapsto g \in qo$.

$Words \mapsto RelW \in wfW$ because $A \mapsto g \in wf$ ($A \mapsto g \in wqo$). If we take a sequence in $sdcW(Words \mapsto RelW)$ the length of words in this sequence decreases then is stationary.

After all words have the same length and with a theorem on cartesian product this sequence cannot decrease infinitely.

We can now use the two theorems on minimal bad sequence. Let bs this minimal bad sequence.

We have $(\forall i \cdot i \geq 0 \Rightarrow bs(i) \neq \varnothing)$ (else bs is not bad) then all words of the bad sequence have a first letter. Let $\{bs(n)(1) | n \geq 0\}$ the set of the first letter of the range of bs.

We can construct the set $Sfix$ of words where the first letter are removed:

$\{w \cdot w \in \mathrm{ran}(bs) | (\lambda i \cdot i \in 1..max(\mathrm{dom}(w)) - 1 | w(i+1))\}$

$\{bs(n)(1) | n \geq 0\}$ is well because include in A

$Sfix$ is well using second theorem on bad sequence

$\{bs(n)(1) | n \geq 0\} \times Sfix$ is also well using cartesian product

$$(\lambda n \cdot n \geq 0 | bs(n)(1)$$
$$\mapsto (\lambda i \cdot i \in 1..max(\mathrm{dom}(bs(n))) - 1 | bs(n)(i+1))$$

is a good sequence and then bs is good also and cannot be bad !

Conclusion

- Quasi-orders, their theorems and proofs are now in Rodin
- 4 weeks of effort
- Most of the proof are the same than those in Cachan lecture (2 littles errors discovered. No tools was used). Higmans and Kruskal follow the Cachan principle but in a more abstract way.
- Thanks to the instantiation plugin. Without it this work will be more difficult, less solid, almost impossible.

Conclusion

Thank you Thierry for introducing me to quasi-orders world

Thank you Jean Goubault-Larrecq for your discussions on this topic

Thanks Guillaume Verdier and Laurent Voisin for the plugin

Conclusion

Thanks Jean-Raymond !