Examples of using the Instantiation Plug-in

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In a companion paper [1], Guillaume Verdier and Laurent Voisin presented a new approach to genericity in the Rodin toolset: this approach is made practical by means of an Instantiation Plug-in. In the present short paper[1] we propose some examples of using this new approach. Note that we constructed more examples: we only present here the most important ones. These examples are preliminary as the plug-in is still under development as stated in [1]. The key to this presentation is to show how such examples can be structured using the Instantiation Plug-in.

Two basic examples are independent: Fixpoint and Wellfoundedness. Other examples depend directly or indirectly of them. This is indicated in the following diagram.

$$\boxed{\begin{array}{ccccc} \mathit{Fixpoint} & \longrightarrow & \mathit{Recursion} & \longleftarrow & \mathit{Wellfoundedness} \\ \downarrow & & & & \downarrow \\ \mathit{Closure} & & & & \mathit{Theorems\ on}\ \mathbb{N} \\ \downarrow & & & & \\ \mathit{Theorems} & & & & \end{array}}$$

1 Fixpoint

Given a set S and a set function h built on S: $h \in \mathbb{P}(S) \rightarrow \mathbb{P}(S)$, a fixpoint of h is a subset $\mathrm{fix}(h)$ of S such that $\mathrm{fix}(h) = h(\mathrm{fix}(h))$. Here is a proposal:

$$\mathrm{fix}(h) \ \hat{=}\ \mathrm{inter}(\{s \mid s \subseteq S \ \wedge\ h(s) \subseteq s\})$$

Assuming that the function h is monotone; we have (Tarski)

$$(\forall a, b \cdot a \subseteq b \ \Rightarrow\ h(a) \subseteq h(b)) \ \Rightarrow\ \mathrm{fix}(h) = h(\mathrm{fix}(h)$$

Moreover $\mathrm{fix}(h)$ is the least fixpoint:

$$\forall t \cdot t = h(t) \ \Rightarrow\ \mathrm{fix}(h) \subseteq t$$

2 Closure

Given a set S and a relation r built on S: $r \in S \leftrightarrow S$, the closure$(r)$ is defined to be the following fixpoint:

$$\mathrm{closure}(r) \ =\ \mathrm{fix}(\lambda s \cdot s \in S \leftrightarrow S \mid r \cup (s \,;\, r))$$

Since we use the fixpoint operator, we have to instantiate (adapt) its definition as provided within the corresponding section. This is exactly what is allowed by the Instantiation Plug-in.

[1] This work is supported by the French ANR project Event-B Rodin Plus (EBRP, ANR-19-CE25-0010)

3 Closure Theorems

We now instantiate the closure definition and prove the following theorems:

$$\text{closure}(r) \;=\; \text{fix}(\lambda s \cdot s \in S \leftrightarrow S \mid r \cup (s \,;\, r))$$

$$\text{closure}(r^{-1}) \;=\; (\text{closure}(r))^{-1}$$

4 Wellfoundedness

We are given a set S and a binary relation r built on S: $r \in S \leftrightarrow S$. If for any x belonging to the range of r, we follow r^{-1} and reach a point which is not in the range of r after a *finite* travel, the relation r is said to be well-founded: $\text{wf}(r)$. It can be given the following formal definition:

$$\text{wf}(r) \;\;\widehat{=}\;\; \forall p \cdot p \subseteq S \;\wedge\; p \subseteq r[p] \;\Rightarrow\; p = \varnothing$$

It can be proved that we have an induction rule for a set with a well-founded relation. Here is the corresponding formal definition:

$$\forall q \cdot q \subseteq S \;\wedge\; (\forall x \cdot x \in S \;\wedge\; r^{-1}[\{x\}] \subseteq q \;\Rightarrow\; x \in q) \;\Rightarrow\; S \subseteq q$$

5 Theorems on Natural Numbers

We now instantiate the set S of previous section to the set $\mathbb{N}$ of natural numbers. It can be proved that the relation "$+1$" on $\mathbb{N}$ is well-founded. As a consequence, we can deduce the classical induction rule for $\mathbb{N}$

6 Recursion

We are given a set S and a well-founded relation r built on S. We are given another set B and a function g defined as follows: $g \in (S \times (S \nrightarrow B)) \nrightarrow B$. Then there exists a unique total function f from S to B with the following property:

$$\forall x \cdot x \in S \;\Rightarrow\; f(x) = g(x \mapsto r^{-1}[\{x\}] \lhd f)$$

This function is defined by means of a fixpoint. We have thus to instantiate the definition of wellfoundedness and that of fixpoints defined in earlier sections.

7 Conclusion

All proofs alluded in this paper were successfully performed using the Instantiation Plug-in and the Rodin Tool.

References

1. Guillaume Verdier and Laurent Voisin: *Context Instantiation Plug-in: a new approach to genericity in Rodin*